Spatial Asymptotic Behavior in PDE and its Application in Finite Element Method

Karen Yeressian[1],

[1]University of Zurich, karen.yeressian@math.uzh.ch

1 Introduction

Many problems in PDE are naturally given in relatively big domains but usually we are interested only in the solution in a sub-domain. So it is natural to study the behavior of the solution in a sub-domain as the size of the whole domain grows.

A very practical problem in this direction is when we want the finite element approximation of the solution in a sub-domain. Then it is natural to triangulate the domain a priori finer in this region of interest and to coarsen the triangulation by getting far from this region of interest.

In this talk we present an iteration technique to obtain estimates that suggest the local behavior of the solution and also the a priori mesh adaptation for the region of interest.

2 Results

We bring two cases of asymptotic behavior estimates. First, we state the local behavior of the solution to an elliptic variational inequality with constraint on the gradient. Second, we bring estimates which suggest the appropriate a priori mesh adaptation for the region of interest in finite element calculations.

2.1 Elliptic Variational Inequality

Let us consider the domain $\omega \subset \mathbb{R}^{d-1}$ and the cylinder $\Omega_\ell = (-\ell, \ell) \times \omega \subset \mathbb{R}^d$. Let us consider in $\mathbb{R}^d$ the norm $|x|_p^p = |x_1|^p + \ldots + |x_d|^p$ and for $g \in H^1(\Omega_\ell)$ such that $g = 0$ on the lateral boundary $(-\ell, \ell) \times \partial\omega$ let us define the closed convex set

$$K_g = \left\{ v \in H^1(\Omega_\ell) \,\middle|\, v - g \in H_0^1(\Omega_\ell),\ |Dv|_p \leq 1 \text{ a.e in } \Omega_\ell \right\}$$

then for $f \in H^{-1}(\Omega_\ell)$ we may consider the variational inequality

$$\begin{cases} \displaystyle\int_{\Omega_\ell} Du \cdot D(u - v) \leq \langle f, u - v \rangle\,, \quad \forall v \in K_g \\ u \in K_g \end{cases}$$

As an example let us consider the case when $d = 2$, $\omega = (0, 1)$, $f = 0$ and $g(x_1, x_2) = \frac{1}{2} - |x_2 - \frac{1}{2}|$ then the solution to the variational inequality for $p = 2$ is depicted in figure **??** for $\ell = 1$ and in figure **??** for $\ell = 1.25$. As the figures suggest the solution converges to 0 in the middle of the cylinder as the length

(a)(b)
$\ell =\ell =$
1 1.25

Figure 1: Solution of Variational Inequality with $p = 2$.

of the cylinder grows. For a harmonic function with the same boundary values this convergence is well known.

For the case $1 < p \leq \infty$ in the talk we present an estimate which when $f = 0$ is

$$\int_{\Omega_{\ell_1}} |Du|^2 \leq Ce^{-\alpha(\ell_2-\ell_1)} \int_{\Omega_{\ell_2}} |Du|^2$$

where C and α are positive constant depending on p. Here α converges to 0 as p converges to 1. The proof is based on an iteration technique.

2.2 A Priori Mesh Adaptation for Region of Interest

Let us consider the polygonal domain $\Omega \in \mathbb{R}^d$ and the problem

$$\begin{cases} -\triangle u + u = f \text{ in } \Omega \\ \dfrac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega \end{cases}$$

let us assume that one is interested in the solution only in a polygonal sub-domain $\Omega' \subset \Omega$, such that $\mathrm{dist}(\partial\Omega, \Omega') >> 1$. To compute the solution by finite element method it is natural to have the finest refinement of the triangulation in the domain Ω' and gradually the triangulation to get coarse away from this region of interest, in this way one tries to obtain less error in the region of interest with the same number of total bases.

Let us fix a sequence of growing polygonal domains

$$\Omega' = \Omega_1 \subset \Omega_2 \subset \cdots \subset \Omega_\ell = \Omega$$

such that

$$d_i = \mathrm{dist}(\Omega_{i-1}, \Omega \backslash \Omega_i) > 0$$

Now let us consider a triangulation of the domain Ω which is compatible with the domains Ω_i in the sense that each Ω_i is a union of triangles in this triangulation. Let us consider the P_1 finite element method.

With similar iteration technique as the problem in the previous subsection was dealt with, we obtain the following local error estimate for the region of interest. There exists constants $\alpha, C > 0$ which depend on the regularity of the triangulation such that denoting by $\hat{u}$ the finite element solution, for any w in our finite element subspace we have the following local error estimate

$$\|u - \hat{u}\|^2_{H^1(\Omega_1)} \leq C\Big(e^{-\alpha \sum_{k=1}^{\ell} \tilde{d}_k} \|u - \hat{u}\|^2_{H^1(\Omega)} + \sum_{i=1}^{\ell} \{\sum_{j=i}^{\ell} e^{-\alpha \sum_{k=1}^{j} \tilde{d}_k}\} \|u - w\|^2_{H^1(\Omega_i \backslash \Omega_{i-1})}\Big)$$

here $\tilde{d_k} = \min(1, d_k)$. We will show that this estimate suggests an appropriate a priori mesh adaptation to achieve less error with a fixed number of bases.

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